Harmonic oscillator under Lévy noise: Unexpected properties in the phase space

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A harmonic oscillator under influence of the noise is a basic model of various physical phenomena. Under Gaussian white noise the position and velocity of the oscillator are independent random variables which are distributed according to the bivariate Gaussian distribution with elliptic level lines. The distribution of phase is homogeneous. None of these properties hold in the general Lévy case. Thus, the level lines of the joint probability density are not elliptic. The coordinate and the velocity of the oscillator are strongly dependent, and this dependence is quantified by introducing the corresponding parameter ("width deficit"). The distribution of the phase is inhomogeneous and highly nontrivial.

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I. INTRODUCTION

Many situations in natural sciences can be successfully investigated adopting the stochastic level of description, considering the system of interest as a dynamical system responding to external perturbations represented by a noise. In the simplest situations this noise is assumed to be white and Gaussian. The white type of the noise is the consequence of the large number of independent interactions bounded in time. Its Gaussian character arises due to the assumption that the interactions are bounded in their strength. In many cases far from equilibrium the second assumption fails; the noise still can be considered as white but is now described by heavy-tailed distributions, often of the α-stable Lévy type [1]. Such heavy-tailed fluctuations are abundant in turbulent fluid flows [2], magnetized plasmas [3], optical lattices [4], heartbeat dynamics [5], neural networks [6], search on a folding polymers [7], animal movement [8], climate dynamics [9], financial time series [10], and even in spreading of diseases and dispersal of banknotes [11]. These observations indicate the need for closer examination of properties of stochastic systems under such heavy-tailed noises.

A damped harmonic oscillator under influence of external noise is a basic model of non-equilibrium statistical physics. The evolution of the state variable x is described by the following equation of motion

$$\ddot{x}(t) = -\gamma\dot{x}(t) - kx(t) + \xi(t) \tag{1}$$

(the mass of the oscillating particle is taken to be unity). Since the response function of the harmonic oscillator is known, the coordinate $x(t)$ and the velocity $v(t) = \dot{x}(t)$ are obtained by the action of a known linear operators on the noise term $\xi(t)$, and the characteristic function of the joint probability distribution of (x, v) can be obtained explicitly [12] by generalizing the results by Doob [13]. However, although the formal solution of the problem is known for almost 30 years, it doesn't seem that the properties of such a basic system have attracted any attention of physicists or mathematicians, although they are astonishingly different from the ones under Gaussian noise. Here, we focus on the properties of the corresponding stationary distribution, which is achieved at times $t \gg \gamma/k$.

In the Gaussian case the distribution of variables (x, v) is a bivariate normal of the type

$$p(x, v) = \frac{1}{2\pi\sigma_x\sigma_v} \exp\left[-\frac{x^2}{2\sigma_x^2} - \frac{v^2}{2\sigma_v^2}\right] \tag{2}$$

where σ_x and σ_v are the widths of the distributions of the coordinate and velocity [14]. Since this distribution factorizes into a product of x- and v-distributions, the phase variables are independent. The independence of x and v carries over to the equipartition theorem of equilibrium statistical physics. Since the argument of the exponential is a quadratic form of the coordinate and velocity, the contours of equal probability density are elliptic (the distribution is *elliptically contoured*, or simply *elliptic*). Introducing the rescaled variables $\tilde{x} = x/\sigma_x$ and $\tilde{v} = v/\sigma_v$, one defines the phase angle $\phi = \arctan(\tilde{v}/\tilde{x})$ which is uniformly distributed over $[-\pi/2, \pi/2)$.

In the case of Lévy noises other than the Gaussian one none of these properties holds. Thus, the variables x and v are dependent, and the dependence is stronger in the overdamped case ($\sqrt{k} < \gamma/2$) than in the underdamped one ($\sqrt{k} > \gamma/2$). The distribution of (x, v) is not elliptic (see Fig. 1), and the phase angle shows an inhomogeneous, highly nontrivial distribution (see Fig. 3). In what follows we present the results of numerical simulations of the harmonic oscillator under Lévy noise together

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with analytical results corroborating these findings. We moreover discuss in some detail the measures which can be used to characterize dependence of the nonelliptic Lévy variables and quantify the dependence of x and v-variables.

II. GENERAL CONSIDERATIONS

Although the initial part of our calculations does not go beyond [12], we reproduce part of the considerations in a physical language for the sake of readability. The formal solution of Eq. (1) is

$$x(t) = F(t) + \int_{-\infty}^{t} G(t-t')\xi(t')dt', \tag{3}$$

where $G(t)$ is the Green's (response) function of the corresponding process, and $F(t)$ is a decaying function (a solution of the homogeneous equation under given initial conditions). The solution for v is given by

$$v(t) = F_v(t) + \int_{-\infty}^{t} G_v(t-t')\xi(t')dt', \tag{4}$$

where $G_v(t)$ is the Green's function of the velocity process

$$G_v(t) = \frac{d}{dt}G(t). \tag{5}$$

In a stationary situation, $t \to \infty$, the F-functions in Eqs. (3) and (4) vanish. The Green's function of Eq. (1) can be easily found e.g. via the Laplace representation, and reads:

$$G(t) = \frac{\exp(-\gamma t/2)}{\sqrt{\omega^2 - \gamma^2/4}} \sin\left[\sqrt{\omega^2 - \gamma^2/4}\; t\right] \tag{6}$$

for $\omega = \sqrt{k} > \gamma/2$ (underdamped case),

$$G(t) = t\exp(-\gamma t/2) \tag{7}$$

for $\omega = \gamma/2$ (critical case) and

$$G(t) = \frac{\exp(-\gamma t/2)}{\sqrt{\gamma^2/4 - \omega^2}} \mathrm{sh}\left[\sqrt{\gamma^2/4 - \omega^2}\; t\right] \tag{8}$$

for $\omega < \gamma/2$ (overdamped case). Note that the functions $G(t)$ vanish both for $t = 0$ and for $t \to \infty$ so that

$$\begin{aligned}\int_0^\infty G(t)\left[\frac{d}{dt}G(t)\right]dt &= \frac{1}{2}\int_0^\infty \left[\frac{d}{dt}G^2(t)\right]dt \\ &= \frac{1}{2}\, G^2(t)\big|_0^\infty = 0,\end{aligned} \tag{9}$$

i.e. $G(t)$ and $G_v(t)$ are orthogonal on $[0,\infty)$.

Let us now confine ourselves to the finite (but long enough) interval of integration and consider integral sums corresponding to Eqs. (3) and (4) for a given time step Δt:

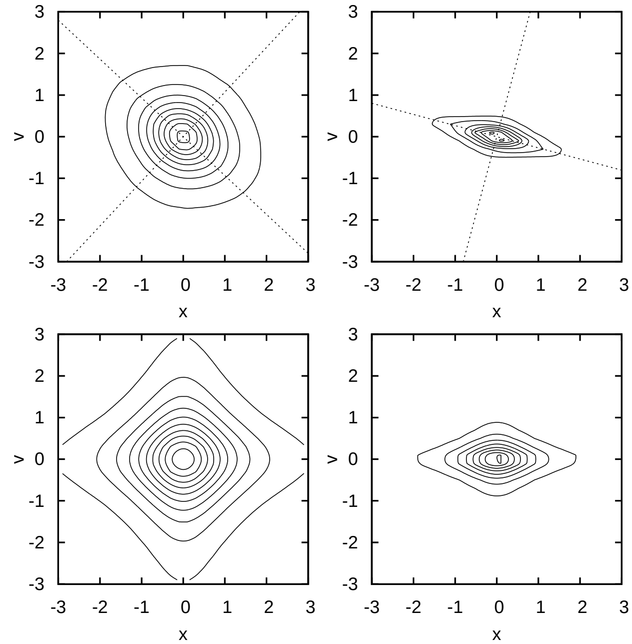

FIG. 1. Joint probability density $p(x, v)$ (top panel) and product of two marginal densities $p_x(x)p_v(v)$ (bottom panel) for $\alpha = 1$. The left panel corresponds to the underdamped case ($k = 1, \gamma = 1$) while the right one to the overdamped case ($k = 1, \gamma = 4$). Dashed lines in the top panel represent major axes described in the text. Simulation parameters: number of repetitions $N = 10^7$, time step of integration $\Delta t = 10^{-2}$.

$$x = \sum_{i=1}^{N} a_i \xi_i \tag{10}$$

and

$$v = \sum_{i=1}^{N} b_i \xi_i \tag{11}$$

with ξ_i being independent identically distributed random variables corresponding to the integrals of the noise over independent Δt-intervals, and a_i, b_i being the values of the Green's functions evaluated at corresponding points.

Let us evaluate the joint PDF of x and v

$$p(x,v) = \int \dots \int \delta\left(x - \sum_{i=1}^{N} a_i\xi_i\right) \delta\left(v - \sum_{i=1}^{N} b_i\xi_i\right) \prod_{i=1}^{N} p(\xi_i)d\xi_i. \tag{12}$$

The characteristic function of this distribution is given

by the Fourier transform

$$
\begin{aligned}
f(k,q) &= \iint e^{ikx+iqv} p(x,v)dxdv \qquad (13)\\
&= \prod_{i=1}^{N} \int \exp[(ika_i + iqb_i)\xi_i] p(\xi_i) d\xi_i \\
&= \prod_{i=1}^{N} f(ka_i + qb_i),
\end{aligned}
$$

where $f(k) = \int e^{ik\xi} p(\xi) d\xi$ is the characteristic function of the distribution of ξ.

In the case considered here, the noise corresponds to a formal derivative of a Lévy random process $L(t)$. The integrals of the noise over finite time intervals are independent Lévy stable random variables. The probability distributions corresponding to such random variables, stable infinitely divisible laws, are limiting distributions in the space of probability densities due to generalized central limit theorems [1, 15]. The characteristic function of $\xi(\Delta t)$ is thus given by

$$
\langle e^{ik\xi(\Delta t)} \rangle = \exp\left[-\Delta t \sigma^\alpha |k|^\alpha \left(1 - i\beta \mathrm{sgn} k \tan \frac{\pi\alpha}{2} \right) \right], \qquad (14)
$$

where $\alpha \in (0,2]$ is the stability index, $\beta \in [-1,1]$ is the asymmetry (skewness) parameter, while σ ($\sigma > 0$) is a scale parameter. Increments of the Lévy motion are distributed according to a α-stable density, which for $\alpha < 2$ has power law asymptotics, i.e. $p(\xi) \propto 1/(\xi)^{\alpha+1}$. The Gaussian distribution is a special case of α stable density with $\alpha = 2$, which is the only one α-stable density possessing finite moments [1]. In our work here we concentrate on the case of symmetric distributions, $\beta = 0$. The characteristic functions of symmetric Lévy distributions have the form

$$
f(k) = \exp(-\Delta t \sigma^\alpha |k|^\alpha). \qquad (15)
$$

Therefore

$$
\prod_{i=1}^{N} f(ka_i + qb_i) = \exp\left[-\Delta t \sigma^\alpha \sum_{i=1}^{N} |ka_i + qb_i|^\alpha \right], \qquad (16)
$$

or returning to the integral notation,

$$
f(k,q) = \exp\left[-\sigma^\alpha \int_0^\infty |kG(t) + qG_v(t)|^\alpha dt \right]. \qquad (17)
$$

The characteristic functions of the marginal distributions are obtained by putting $q = 0$ (for x-distribution) or $k = 0$ (for v-distribution). Note moreover, that the characteristic function of the distribution of any linear combination of x and v

$$
z = ax + bv \qquad (18)
$$

will read

$$
f_z(k) = f(ak, bk) \qquad (19)
$$

and is the one of the symmetric Lévy distribution, so that the distribution of x and v is a *bona fide* bivariate stable one. It is however not of elliptic type, for which case the characteristic function would be

$$
f(k,q) = \exp\left[-\sigma^\alpha \int_0^\infty |w_x^2 k^2 + C_{xv} kq + w_v^2 q^2|^{\alpha/2} dt \right] \qquad (20)
$$

i.e. a function of a quadratic form in k and q, see [16].

The characteristic function, Eq. (17) does not decouple into the product of the functions of k and q only, i.e. the distributions of x and v are *not independent* (we shall say, they are *associated*), except for the Gaussian ($\alpha = 2$) case. In this case

$$
\begin{aligned}
\int_0^\infty |kG(t) + qG_v(t)|^2 dt = k^2 &\int_0^\infty G^2(t) dt \qquad (21)\\
+ q^2 &\int_0^\infty G_v^2(t) dt \\
+ 2kq &\int_0^\infty G(t) G_v(t) dt,
\end{aligned}
$$

and the last integral vanishes due to orthogonality of $G(t)$ and $G_v(t)$. In all other cases such decoupling does not take place. The Gaussian case is also the only case when the distribution is elliptic.

In order to demonstrate lack of independence of x and v for the harmonic oscillator under Lévy noise we have estimated from numerical simulations of the process the joint distribution $p(x,v)$ and compared it with product of two marginal densities $p_x(x)p_v(v)$, see Fig. 1, presented both for the underdamped (left panel) and for the overdamped (right panel) case under Cauchy ($\alpha = 1$) noise [17]. The contours of the corresponding joint probability densities are central symmetric, and the positions of their main axes, along which the width is the largest or the smallest, can be obtained numerically by finding the extrema of

$$
w(\theta) = \int_0^\infty |G(t)\cos\theta + G_v(t)\sin\theta|^\alpha dt. \qquad (22)
$$

The corresponding axes are at the angles of approximately -43° (longer axis) and 47° (shorter one) for the underdamped case, and of approximately -15° (longer axis) and 75° (shorter one) for the overdamped case. The corresponding angles are close (but not equal to) the positions of extrema in the phase plot, due to different weighting of the variables, compare Figs. 1 and 3.

III. MEASURES OF DEPENDENCE

The case of dependent random variables lacking moments poses a problem of quantifying the strength of their dependence. In the "normal" Gaussian case, and in every case when the second moments of the variables are finite, the coefficient of covariance gives us a standard measure of such dependence. For variables with zero mean

$$
\mathrm{cov}(x,y) = \langle xy \rangle, \qquad (23)
$$

and for variables whose mean values are nonzero the same is defined for the centered variables $x_c = x - \bar{x}$ and $y_c = y - \bar{y}$. In what follows we consider only variables centered at zero, and omit the subscript c. To characterize the relative strength of the dependence one can introduce the correlation coefficient $\mathrm{corr}(x, y)$ by normalizing the covariance over the dispersions of the corresponding centered variables. Alternatively, one can consider the covariance of variables x and y normalized with respect to their dispersions $\sigma_x = \langle \tilde{x}^2 \rangle^{1/2}$ and $\sigma_y = \langle \tilde{y}^2 \rangle^{1/2}$, i.e. $\tilde{x} = x/\sigma_x$, $\tilde{y} = y/\sigma_y$:

$$\mathrm{corr}(x, y) = \frac{\mathrm{cov}(x, y)}{\sigma_x \sigma_y} = \mathrm{cov}(\tilde{x}, \tilde{y}). \tag{24}$$

The correlation coefficient of two random variables changes in the interval $[-1, 1]$ and it is unity if the variable y is a copy of x up to arbitrary rescaling, and to minus unity if y is a copy of x taken with the opposite sign, again up to the arbitrary change of scale. The correlation coefficient of independent variables vanishes. Since the definition of the correlation coefficient involves the second moment of the corresponding random variables, it cannot be used immediately for the ones lacking such a moment. Here several generalizations have been proposed, all seeming to be deficient for our purpose, consequently we have to extend their definitions.

A relatively common measure of dependence of Lévy variables is the *codifference* [18]. To understand its nature it is enough first to consider the Gaussian case and to turn to the characteristic function of the corresponding bivariate distribution of x and y

$$\begin{aligned} f(k_1, k_2) &= \int\int dx dy e^{ik_1 x + ik_2 y} p(x, y) \\ &= \exp\left[-\sigma_x^2 k_1^2 - \sigma_y^2 k_2^2 - 2\mathrm{cov}(x, y) k_1 k_2\right]. \end{aligned} \tag{25}$$

To assess the covariance one can then consider the difference variable $d = x - y$ whose characteristic function is

$$\begin{aligned} f_d(k) &= f(k, -k) \\ &= \exp\left[-\left(\sigma_x^2 + \sigma_y^2 - 2\mathrm{cov}(x, y)\right) k^2\right], \end{aligned} \tag{26}$$

from which

$$\mathrm{cov}(x, y) = \frac{1}{2}\left[\ln f_d(k)|_{k=1} + \sigma_x^2 + \sigma_x^2\right], \tag{27}$$

or in slightly different notation

$$\mathrm{cov}(x, y) = \frac{1}{2}\mathrm{codiff}(x, y) \tag{28}$$

with $\mathrm{codiff}(x, y)$ being the codifference of the variables x and y defined through

$$\begin{aligned} \mathrm{codiff}(x, y) = &\ln f(k, -k)|_{k=1} \\ &+ \ln f(k, 0)|_{k=1} + \ln f(0, k)|_{k=1}. \end{aligned} \tag{29}$$

The definition of the codifference as a measure of dependence via characteristic functions does not rely on the existence of any moments and is universal, i.e. it can be used in the Lévy case as well. We note that equivalently one can use the sum of the two variables x and y, and to define the "cosum" as

$$\begin{aligned} \mathrm{cosum}(x, y) = &-\ln f(k, k)|_{k=1} \\ &+ \ln f(k, 0)|_{k=1} + \ln f(0, k)|_{k=1} \end{aligned} \tag{30}$$

The codifference (or cosum) as a measure of dependence has two important drawbacks, the first one quite specific to their definitions, and the second one inherited from the the definition of covariance itself.

First, defining the corresponding parameters via characteristic functions, being the means of strongly oscillating exponentials implies numerically generating very large data samples. Therefore, for the sake of practicability, the measure of dependence has to rely on some robust and easily accessible statistics.

Second, using the codifference as a measure of dependence of the observables of different dimension (coordinate and velocity) makes no sense physically, since the change in units of measurement changes the measure of dependence. The codifference measure is often used in the analysis of time series, since the dimensions of all terms are the same, the problem of units therefore do not stand. The same is true also for the normalized codifference [19], which is to no extent a simple analogue of the correlation coefficient.

Let us first address the problem of statistics. For a multivariate Lévy distributions, as the one appearing in the previous Section, both the distributions of the the difference $d = x - y$, $p_d(d)$, and of the sum $s = x + y$, $p_s(s)$, of x and y (in our initial problem the variable y corresponds to the velocity v of the particle) are univariate Lévy distributions. The widths of these distributions are exactly the prefactors of $|k|^\alpha$ in the corresponding characteristic functions

$$w_d^\alpha = |\ln f(k, -k)|_{k=1}| \tag{31}$$

and

$$w_s^\alpha = |\ln f(k, k)|_{k=1}|. \tag{32}$$

Analogously w_x and w_y are equal to

$$w_x^\alpha = |\ln f(k, 0)|_{k=1}| \tag{33}$$

and

$$w_y^\alpha = |\ln f(0, k)|_{k=1}|. \tag{34}$$

Therefore

$$\mathrm{codiff}(x, y) = w_d^\alpha - w_x^\alpha - w_y^\alpha, \tag{35}$$

and

$$\mathrm{cosum}(x, y) = w_s^\alpha - w_x^\alpha - w_y^\alpha. \tag{36}$$

Contrary to the initial definition via characteristic functions, the ones over the widths (scaling parameters)

of the distributions can be easily assessed numerically since they are connected with the robust characteristics like the interquartile distances of the corresponding distributions or with the heights of their probability densities at zero

$$p(0) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-w^\alpha |k|^\alpha} dk = \frac{\Gamma(1/\alpha)}{\pi w}, \quad (37)$$

which are easier to read out from the numerically obtained histograms. This "height measure" is repeatedly used in our work.

To circumvent the second problem, we propose the following measure of dependence D corresponding to the cosum of the normalized variables. Both distributions, the one of x and the one of v are first normalized by introducing $\tilde{x} = x/w_x$ and $\tilde{v} = v/w_y$ (both possessing Lévy distributions of unit width). Then we compare the width of the distribution of $\zeta = \tilde{x} + \tilde{v}$ (raised to the power of α) with 2, the sum of width of the distributions of $\tilde{x}$ and $\tilde{v}$:

$$D = \frac{w_\zeta^\alpha - 2}{2}, \quad (38)$$

and call this measure of dependence the width excess (the letter D stands here for "Dependence"). The measure D, the cosum of rescaled variables, changes in the interval $[-1, 1]$. It is zero for independent variables, equal to 1 if the variable v is a copy of x up to arbitrary rescaling (i.e. up to the change of units) and is equal to -1 if the variable v is a copy of $-x$, up to rescaling. It is clear that D is insensitive to the units in which x and v are measured.

For a bivariate Gaussian density D is exactly the correlation coefficient

$$D = \frac{\langle xv \rangle}{\sigma_x \sigma_v} = \mathrm{corr}(\tilde{x}, \tilde{y}). \quad (39)$$

For the elliptic Lévy process it is connected with the Press' association parameter [16], which however can hardly be generalized to nonelliptic cases and vanishes for isotropic distributions, the ones with characteristic function [18]

$$f(k, q) = \exp\left[-\sigma^\alpha (k^2 + q^2)^{\alpha/2}\right], \quad (40)$$

although the corresponding x and v values are not independent, see Eqs. (38) and (44).

The characteristic function of the distribution of ζ

$$\zeta = \tilde{x} + \tilde{v} = x/w_x + v/w_v$$

is

$$f_\zeta(k) = f(k/w_x, k/w_v).$$

Consequently the width excess is

$$D = \frac{1}{2}\int_0^\infty \left|\frac{G(t)}{I_1} + \frac{G_v(t)}{I_2}\right|^\alpha dt - 1 \quad (41)$$

with

$$I_1 = \left[\int_0^\infty |G(t)|^\alpha dt\right]^{1/\alpha} \quad (42)$$

and

$$I_2 = \left[\int_0^\infty |G_v(t)|^\alpha dt\right]^{1/\alpha}. \quad (43)$$

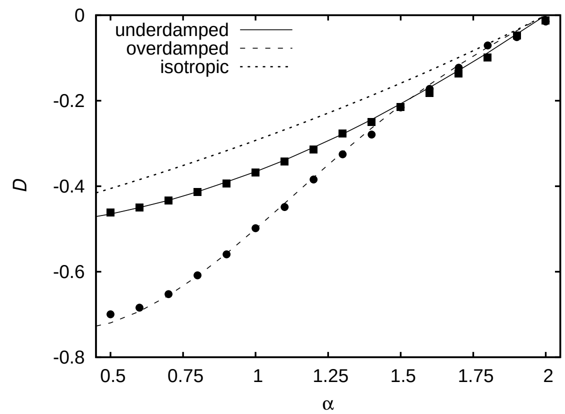

FIG. 2. Width excess D, see Eq. (41), for various α. Lines represent theoretical formulas for underdamped ($k = 1, \gamma = 1$) and overdamped ($k = 1, \gamma = 4$) cases, points correspond to simulation results. The dashed line presents values of the width excess D for isotropic case, see Eqs. (40) and (44). Other simulation parameters as in Fig. 1.

The width excess (41) vanishes for $\alpha = 2$ only, indicating the independence of the position and velocity variables. Contrary to the $\alpha = 2$ case, for $\alpha < 2$ it takes negative values, indicating anti-association, i.e. dependence between the position and the velocity. The results of simulations and calculations of the width deficit for the underdamped and overdamped cases are given in Fig. 2, together with the result for an isotropic case, for which

$$D = 2^{\alpha/2-1} - 1 \quad (44)$$

corresponding to all elliptic distributions, see Eqs. (38) and (40). Figure 2 demonstrates perfect agreement between theoretical results given by formula (41) (lines) and numerical data (points). Negative values of the width excess D indicate strong negative association, which is stronger for the overdamped case ($k = 1, \gamma = 4$) than for the underdamped case ($k = 1, \gamma = 1$).

IV. PHASE AND AMPLITUDE DISTRIBUTIONS

The strong anti-association of velocity and coordinate is reflected in the distribution of the phase angle

$$\phi = \arctan\left[\frac{\tilde{v}}{\tilde{x}}\right] = \arctan\left[\frac{v/w_v}{x/w_x}\right]. \quad (45)$$

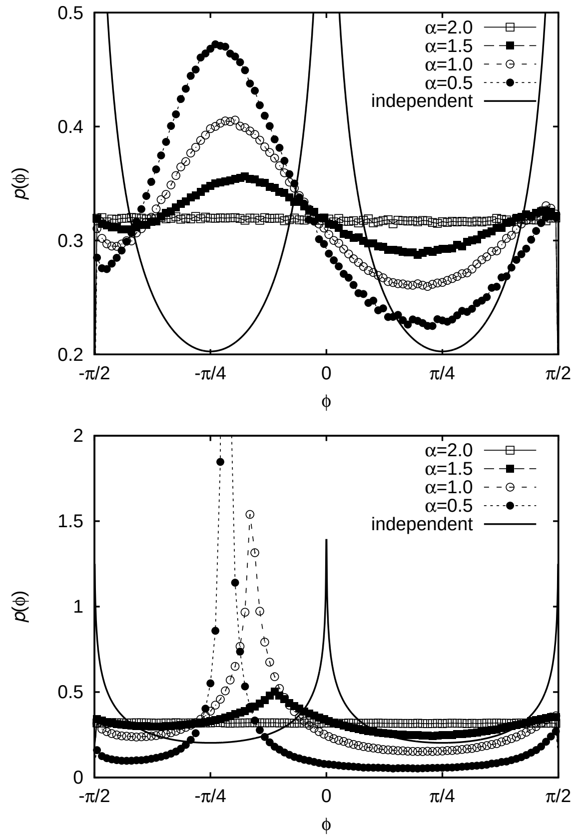

FIG. 3. Phase distribution $p(\phi)$ in the underdamped ($k = 1, \gamma = 1$ – top panel) and overdamped ($k = 1, \gamma = 4$ – bottom panel) cases for different values of α. The full line shows the corresponding distribution for independent Cauchy variables, see Eq. (46). Simulation parameters as in Fig. 1

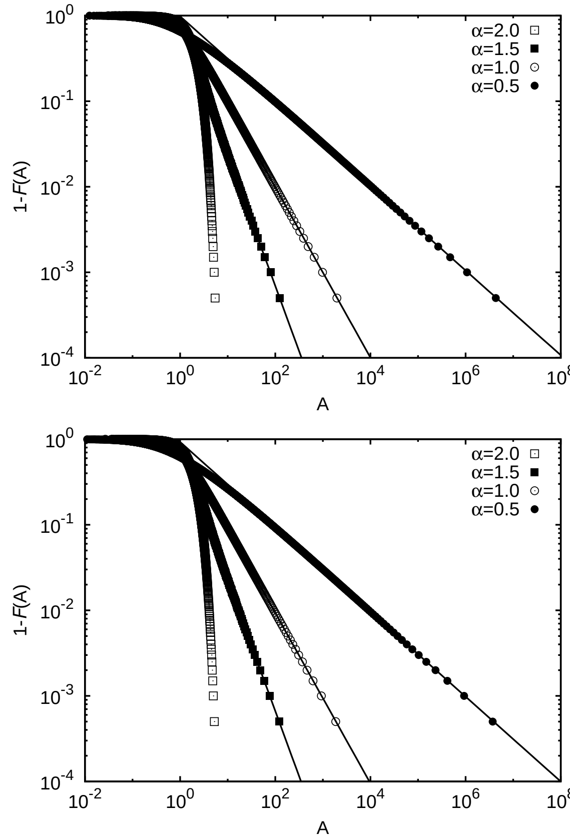

FIG. 4. Complementary cumulative distribution $(1 - F(A))$ of the amplitude $A = \sqrt{\tilde{x}^2 + \tilde{v}^2} = \sqrt{(x/w_x)^2 + (v/w_v)^2}$ in the underdamped ($k = 1, \gamma = 1$ – top panel) and overdamped ($k = 1, \gamma = 4$ – bottom panel) cases for different values of α. Full lines show the corresponding asymptotic power-law decays of complementary cumulative distributions, i.e. $A^{-\alpha}$. Simulation parameters as in Fig. 1

This distribution is homogeneous over $[-\pi/2, \pi/2)$ for all elliptic bivariate Lévy laws (e.g. in the Gaussian case) and has a marked form with peaks (divergences) at $\phi = 0, \pm\pi/2$ for independent random Lévy variables (corresponding to the star-like form of the product of two Lévy distributions, see Fig. 1). The actual forms of the phase PDFs are shown in Fig. 3 for the underdamped (top panel) and overdamped (bottom panel) cases. It is interesting that the peaks are present at "nontrivial" angles, the ones different from 0 and $\pm\pi/2$.

For $\alpha < 2$, strong deviations both from the uniform distribution and from the product distribution are well visible. For $\alpha = 1$ the phase distribution for independent x- and v-variables is given by

$$p(\phi) = \frac{1}{\pi^2 \cos 2\phi} \ln \left[\frac{1 + \cos 2\phi}{1 - \cos 2\phi} \right], \tag{46}$$

see Fig. 3.

The amplitude

$$A = \sqrt{\tilde{x}^2 + \tilde{v}^2} = \sqrt{(x/w_x)^2 + (v/w_v)^2} \tag{47}$$

is asymptotically distributed according to a power-law distribution

$$p(A) \propto \frac{1}{A^{\alpha+1}}, \tag{48}$$

consequently the complementary cumulative distribution has also power-law tails which are characterized by the exponent α, see Fig. 4, as it follows from the results of Ref. [12].

V. SUMMARY

In situations far from equilibrium interactions of the harmonic oscillator with the environment can often be described by the white Lévy noise. Presence of Lévy noises introduces dependence (association) between velocity and position, which vanishes only in the Gaussian limit of $\alpha = 2$. The presence of association between position and velocity is manifested by the nontrivial joint distribution of $p(x, v)$ and nontrivial phase distribution $p(\phi)$. Our main finding corresponds to strong

anti-association between the position and the velocity of a Lévy-driven oscillator, a property which might be of high importance for first passage properties of such a process. For example, in the Kramers problem the dependence of position and velocity might be responsible for the breakdown of the transition-state description. A direct consequence of (anti-)association between velocity and position is the breakdown of equipartition theorem of equilibrium statistical physics for the harmonic Lévy oscillator. The recorded findings, together with results of earlier investigations [1, 20, 21] clearly show that properties of out-of-equilibrium (Lévy noise driven) systems are very different from their equilibrium (Gaussian noise driven) counterparts.

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